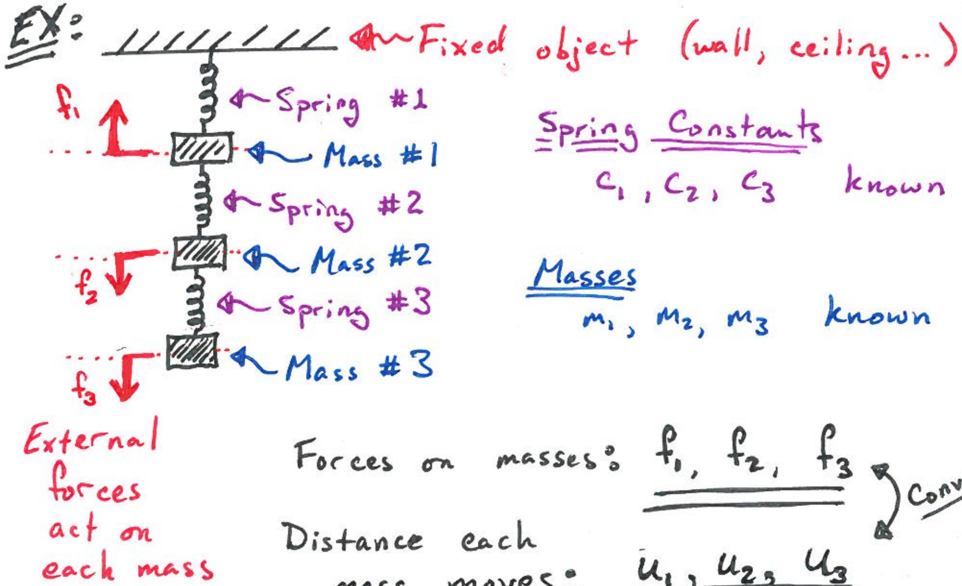
1-Dimensional Spring Systems: Equilibrium and the Stiffness Matrix

Setup:

Given a system of masses attatched to each other with springs (known spring const. and weights). Some extra outside forces act on each mass, causing them to move which stretches/contracts springs.

- Question 1: Calculate the forces required to cause given movements.
- Question 2: Calculate the movements caused by given forces.

EX:



Spring Constants
c_1, c_2, c_3 known

Masses
m_1, m_2, m_3 known

Forces on masses: f_1, f_2, f_3

Convert

Distance each mass moves: u_1, u_2, u_3

To simplify, we break the problem into three related processes:

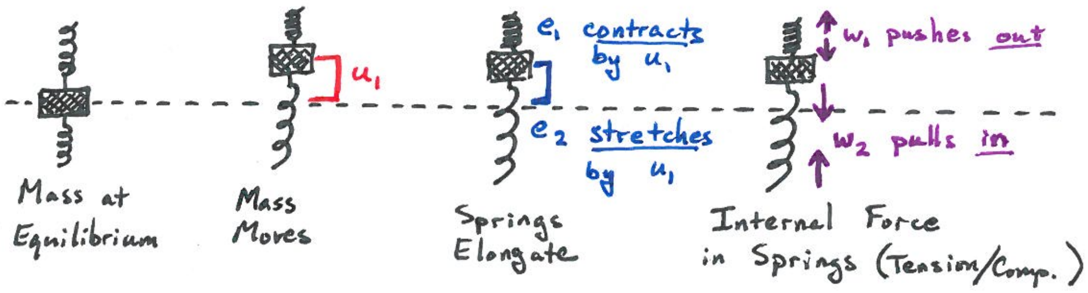
(Imagine grabbing each mass and pushing it to a new position. We will will calculate the "balancing force" required to hold the masses in place away from equilibrium.)

Process #1. Displacement of masses causes Elongation of springs $\quad u \downarrow e$

(Positive elongation = stretching
Negative elongation = contracting)

Process #2. Elongation of springs causes Internal Force in each spring $\quad e \downarrow w$

Process #3. Internal Force in springs pull/push masses at each end. These combined forces must be balanced by a Force at each mass. $\quad w \downarrow f$

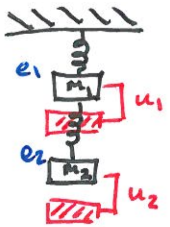


(The force required to hold the mass must balance both the force w_1 & w_2.)

Each process can be written as a matrix operation.

Notation: Usually we imagine the spring system hanging, with gravity pulling each mass downward.
→ Down will be the positive direction.

Process #1. $e = Au$ A is "elongation matrix"



If mass 1 moves down by u_1
spring 1 stretches by u_1
spring 2 contracts by u_1

If mass 2 moves down by u_2
spring 2 stretches by u_2

Total change: $\begin{cases} e_1 = u_1 \\ e_2 = u_2 - u_1 \end{cases}$ → $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{\text{"Elongation Matrix" } A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Note: e gives change in distance between masses.

The elongation matrix A is a "node-edge incidence matrix" for a "directed network."

→ It can be viewed in two ways

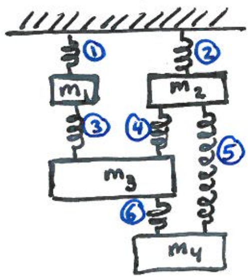
by row or by column

[A has a row for each spring and a column for each mass] $e = Au$

→ The row for a spring has
- 1 marking the mass at the bottom of spring
- -1 marking the mass at the top of spring
- 0 for masses not attached to spring

→ The column for a mass has
- 1 marking springs attached to top of mass
- -1 marking springs attached to bottom of mass
- 0 for springs not attached to mass

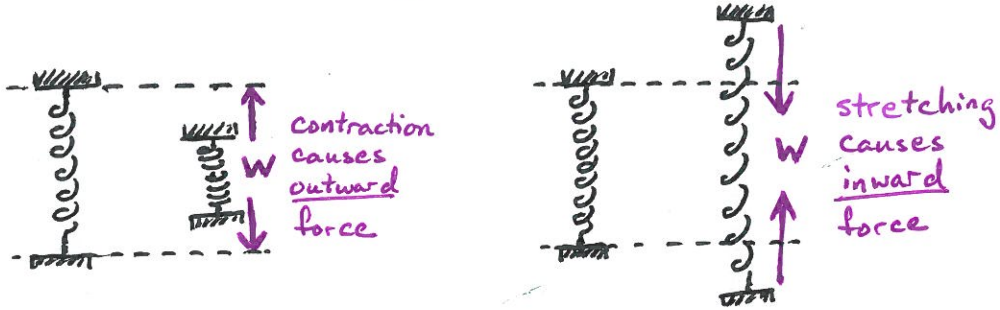
EX:



$$A = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{l} \text{spring 1} \\ \text{spring 2} \\ \text{spring 3} \\ \text{spring 4} \\ \text{spring 5} \\ \text{spring 6} \end{array} \\ \begin{array}{cccc} \text{Mass 1} & \text{Mass 2} & \text{Mass 3} & \text{Mass 4} \end{array} & \end{array}$$

Process #2. $w = Ce$ C is "spring constant" matrix

Elongation of a spring causes an internal force in spring in opposite direction.



Note: Masses on each end of spring will feel force in opposite directions.

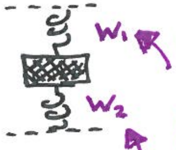
(Internal force) = (spring const.) · (elongation)

$$w_k = c_k \cdot e_k$$

$$\underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 & \cdots \\ 0 & c_2 & 0 & \cdots \\ 0 & 0 & c_3 & \\ \vdots & \vdots & & \ddots \end{bmatrix}}_{C} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{bmatrix}$$

C (diagonal matrix of spring constants.)

Process #3. $f = A^T w$ combine forces at each mass



w_1 { outward force (contraction) above mass pushes down; inward force (stretching) above mass pulls up }

w_2 { forces below mass are reverse. }

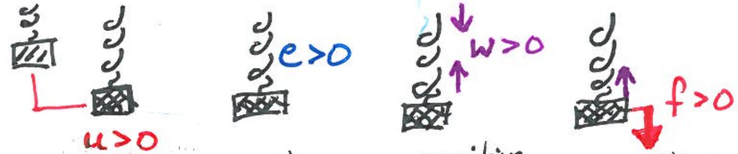
[Total force at mass is given by adding the forces from all springs attatched to the mass.]

Total force: $\{ f_1 = w_1 - w_2 \} \rightarrow [f_1] = [1 \ \ -1] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

(opposite sign because the spring below the mass pulls in opposite direction.)

Note: Closely following signs, you will discover that we are calculating

(Balancing Force) = −(Total Force)



positive (downwards) movement	positive (stretching) elongation above	positive (inwards) internal force	positive (downwards) balancing force required below spring

The matrix combining internal forces in springs to compute balancing force at masses has

- **row** for each mass (balancing force)
- **column** for each spring (internal force)

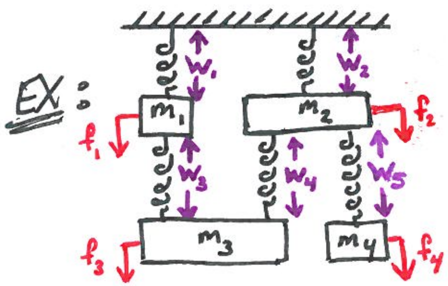
→ The **row** for a mass has

$$\begin{cases} 1 & \text{marking springs attached to top} \\ -1 & \text{marking springs attached to bottom} \\ 0 & \text{for springs not attached to mass} \end{cases}$$

This is an "edge-node incidence matrix"

> … This matrix is the **transpose** of the elongation matrix from earlier!

EX:



$$\begin{cases} f_1 = w_1 - w_3 \\ f_2 = w_2 - w_4 - w_5 \\ f_3 = w_3 + w_4 \\ f_4 = w_5 \end{cases}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix}$$

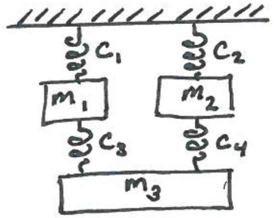
(This is the **transpose** of the matrix on page 2)

→ Combining all of these gives the "**Stiffness Matrix**" K

relating **motions** of masses to **force** on masses.

$$f = A^T \overbrace{C \underbrace{A u}_{e}}^{w}$$

EX:



$$\left.\begin{aligned} c_1 &= 3 \\ c_2 &= 1 \\ c_3 &= 2 \\ c_4 &= 4 \end{aligned}\right\} \text{Spring Constants}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

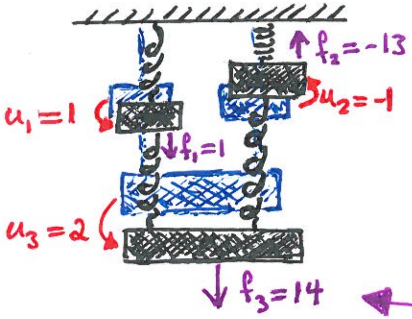
$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3+2 & 0 & -2 \\ 0 & 1+4 & -4 \\ -2 & -4 & 2+4 \end{bmatrix}}_{K} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

"Stiffness Matrix"

(EX continues...)

To hold the spring system in position $\begin{cases} u_1 = 1 \\ u_2 = -1 \\ u_3 = 2 \end{cases}$



requires force:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 5 & -4 \\ -2 & -4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 \\ -13 \\ 14 \end{bmatrix}}}$$

Alternately...

Forces on masses $\begin{cases} f_1 = 6 \\ f_2 = -3 \\ f_3 = 12 \end{cases}$ cause movement:

$$\begin{bmatrix} 5 & 0 & -2 \\ 0 & 5 & -4 \\ -2 & -4 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

(... Solve using LU-Decomposition...)

The stiffness matrix is actually <u>very easy</u> to calculate.

K = "<u>Node-Node Adjacency Matrix</u>" for a weighted, directed network.

→ We can write K <u>without</u> computing A, A^T, or C !!

Given a spring system with spring constants $c_1, c_2, c_3, \ldots$ etc. The <u>stiffness matrix</u> K for the system has

- a <u>row</u> for each <u>mass</u>
- a <u>column</u> for each <u>mass</u>

The matrix K has two types of elements

- Diagonal terms
- Off-diagonal terms

<u>Diagonal</u>. Each mass has one diagonal term.

The diagonal term is equal to the <u>sum</u> of the spring constants of all springs connected to the mass.

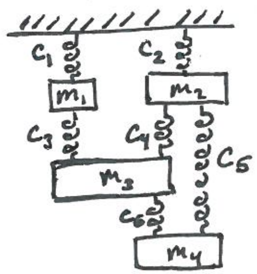
<u>Off-Diagonal</u>.

The non-diagonal terms mark which masses are connected by springs.

0 if no spring between masses

$-c_k$ if a spring with const. c_k connects them

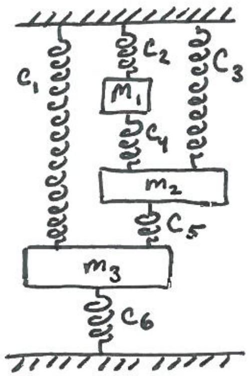
<u>EX:</u>



<u>Stiffness Matrix</u>

$$K = \begin{bmatrix} c_1 + c_3 & 0 & -c_3 & 0 \\ 0 & c_2 + c_4 + c_5 & -c_4 & -c_5 \\ -c_3 & -c_4 & c_3 + c_4 + c_6 & -c_6 \\ 0 & -c_5 & -c_6 & c_5 + c_6 \end{bmatrix}$$

EX: Compute the force required to hold the given spring system at displacements



$$u_1 = 2, \quad u_2 = -1, \quad u_3 = 1$$

$$c_1 = 2, \quad c_2 = 1, \quad c_3 = 5, \quad c_4 = 3$$
$$c_5 = 1, \quad c_6 = 4$$

(Stiffness Matrix)

$$\begin{array}{c|ccc|} \text{mass 1} & c_2 + c_4 & -c_4 & 0 \\ \text{mass 2} & -c_4 & c_3 + c_4 + c_5 & -c_5 \\ \text{mass 3} & 0 & -c_5 & c_1 + c_5 + c_6 \\ & \text{mass 1} & \text{mass 2} & \text{mass 3} \end{array}$$

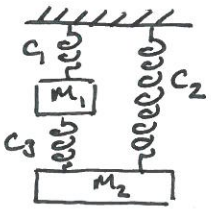
$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 9 & -1 \\ 0 & -1 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -12 \\ 12 \end{bmatrix}$$

Force on mass 1 = 9

Force on mass 2 = -12

Force on mass 3 = 12

EX: Compute the displacement resulting from the following forces



$$f_1 = -2 \qquad f_2 = 3$$

$$c_1 = 3 \qquad c_2 = 1 \qquad c_3 = 2$$

(Stiffness Matrix)

$$K = \begin{bmatrix} c_1 + c_3 & -c_3 \\ -c_3 & c_2 + c_3 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

2x2 inverse

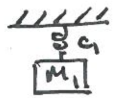
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{19} \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 11/19 \end{bmatrix}$$

Mass 1 does not move.

Mass 2 moves down by 11/19.

EX Compute elongation & stiffness matrices

A)



$$A = [1] \qquad K = [c_1]$$

B)

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad K = [c_1 + c_2]$$